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Eddy viscosity of three-dimensional flow

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Detailed theoretical and numerical results are presented for the eddy viscosity of three-dimensional forced spatially periodic incompressible flow.

As shown by Dubrulle & Frisch (1991), the eddy viscosity, which is in general a fourth-order anisotropic tensor, is expressible in terms of the solution of auxiliary problems. These are, essentially, three-dimensional linearized Navier–Stokes equations which must be solved numerically.

The dynamics of weak large-scale perturbations of wavevector $\boldsymbol{k}$ is determined by the eigenvalues – called here 'eddy viscosities' – of a two by two matrix, obtained by contracting the eddy viscosity tensor with two $\boldsymbol{k}$-vectors and projecting onto the plane transverse to $\boldsymbol{k}$ to ensure incompressibility. As a consequence, eddy viscosities in three dimensions, but not in two, can become complex. It is shown that this is ruled out for flow with cubic symmetry, the eddy viscosities of which may, however, become negative.

An instance is the equilateral *ABC*-flow ($A = B = C = 1$). When the wavevector $\boldsymbol{k}$ is in any of the three coordinate planes, at least one of the eddy viscosities becomes negative for $R = 1/\nu > R_c \simeq 1.92$. This leads to a large-scale instability occurring for a value of the Reynolds number about seven times smaller than instabilities having the same spatial periodicity as the basic flow.

1. Introduction

After Navier (1823) had shown that – at that time hypothetical – molecular motion leads to a viscous diffusion term in the equation for fluid motion, de Saint Venant (1851) observed that flow in wide channels possesses complex eddy motion which considerably enhances the 'friction coefficient' (as the viscosity was called at the time). Boussinesq (1870) proposed a formula for what is now called the eddy viscosity, namely

$$\nu_E = A u_0 h, \tag{1}$$

where A is a dimensionless constant, u_0 is a typical speed and h is a typical scale (e.g. the radius of curvature for pipe flow). Only after the work of Taylor (1915) and Prandtl (1925) did such a 'mixing length' expression of the eddy viscosity become of widespread use in the modelling of turbulent flow.

It was realized quite early that there is a strong analogy between, on the one hand, microscopic transport (the effect of molecular motion and collisions on scales much larger than the mean free path) and, on the other hand, turbulent transport (see Lamb 1916).

The systematic derivation of the hydrodynamical equations from kinetic theory uses singular perturbation techniques in which the small parameter is the Knudsen number (the mean free path λ divided by the hydrodynamical scale l_0). This goes back to the works of Hilbert, Chapman and Enskog (see e.g. Brush 1986, chapter 12).

Similarly, when considering macroscopic fluid motion on scales $\sim l_0$, in order to derive a transport equation with an eddy viscosity, a necessary condition is that the transport should take place on hyperscales $L \gg l_0$. This condition is, however, far from sufficient. Indeed, microscopic and macroscopic motion are of very different nature, the former being conservative while the latter are dissipative. Consequently, macroscopic motion will decay unless constantly regenerated.

Here, we shall assume that the motion is regenerated by a driving force which is periodic or random in space and time, a simple instance of which is the Kolmogorov flow (Meshalkin & Sinai 1961; Nepomnyashchy 1976; Sivashinsky 1985; Bondarenko, Gak & Dolzhansky 1979).

As is well-known, the form of the ordinary hydrodynamical equation is dictated to a large extent by the conservation laws and symmetries (invariance group) of microscopic motion. Similarly, the form of the hyperscale equations depends on the assumed symmetries of the force. Using a multiscale technique, Dubrulle & Frisch (1991, referred to hereinafter as DF) showed that if the force is parity-invariant (possesses a centre of symmetry), the hyperscale behaviour is formally diffusive, that is, governed by linear partial differential equations with first-order time derivatives and second-order space derivatives. In the absence of parity-invariance, first-order space derivatives, leading to the so-called AKA-effect, may also be present (Frisch, She & Sulem 1987; Sulem *et al.* 1989). One way in which eddy viscosities may differ dramatically from molecular viscosities is that they need not be positive. Vergassola (1993) and Gama, Vergassola & Frisch (1994) have investigated the case of two-dimensional flow and found that it frequently has negative eddy viscosity.

For three-dimensional flow, which is our main concern here, eddy viscosities have been calculated so far only in special instances: flow δ-correlated in time (Gama *et al.* 1994, Appendix D) and low-Reynolds number flow (DF). The difficulty is that, in general, numerical calculation of the eddy viscosity tensor is unescapable. This feature is present in most multiscale problems, e.g. the heat equation with periodic coefficients in more than one dimension (Bensoussan, Lions & Papanicolaou 1978).

The paper is organized as follows. In §2.1 we briefly recall the multiscale machinery for determining eddy viscosities. This section may be skipped by readers already familiar with DF. In §2.2 we give a new definition of the eddy viscosity as an eigenvalue of a suitable operator. Section 3 is devoted to the calculation of eddy viscosities in three dimensions, mostly by numerical methods (§3.1). Section 4 is devoted to applications: complex eddy viscosities (§4.1, already briefly reported in Wirth 1994) and negative eddy viscosity instability for the *ABC*-flow (§4.2). Section 4.2 also includes some remarks about the thermodynamics of negative (eddy) viscosity. This issue of isotropy for the eddy viscosity is discussed in §5.

2. Theory of three-dimensional eddy viscosities

2.1. *Multiscale technique for eddy viscosities*

Our purpose is to show that a small-scale flow, produced by a prescribed parity-invariant driving force, modifies the diffusion of momentum at large scales: the molecular viscosity is changed into an eddy viscosity, which is usually a tensor.

The starting point is the three-dimensional Navier–Stokes equation for flow subject to space- and time-periodic forcing. In the notation of DF it reads:

$$\left.\begin{aligned} \partial\cdot\boldsymbol{u} &= 0, \\ \partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\partial)\,\boldsymbol{u} &= -\partial p+\nu\,\partial^2\boldsymbol{u}+\boldsymbol{f}. \end{aligned}\right\} \tag{2}$$

The solution $\boldsymbol{u}$ and the force f are all supposed to be 2π-periodic in t, x_1, x_2, x_3 and to have vanishing space–time averages. The formalism can, in principle, be extended to quasi-periodic or random space–time dependence. This requires significant changes in the numerical methods and will not be discussed here.

The basic flow $(p, \boldsymbol{u})$ is now subject to a weak perturbation:

$$p \to p + \eta P, \quad \boldsymbol{u} \to \boldsymbol{u} + \eta \boldsymbol{W}, \tag{3}$$

where η denotes a small parameter. By omitting the $O(\eta^2)$-terms we get the following linearized Navier–Stokes equation, which is here linearized around the basic flow $(p, \boldsymbol{u})$:

$$\left.\begin{aligned} \partial \cdot \boldsymbol{W} &= 0, \\ \partial_t W_i + \partial_j (u_i W_j + u_j W_i) &= -\partial_i P + \nu \partial^2 W_i. \end{aligned}\right\} \tag{4}$$

This is a partial differential equation with periodic coefficients. The large-scale, long-time behaviour of the perturbation is derived using multiscale analysis. The solution $(P, \boldsymbol{W})$ is assumed to depend both on a fast variable $\boldsymbol{x}$, corresponding to the basic flow and on a slow variable $\boldsymbol{X} = \epsilon \boldsymbol{x}$, where ϵ (the scale ratio) tends to zero. (Since only the linear large-scale dynamics is needed for the calculation of the eddy viscosity, the limit $\eta \to 0$ may be taken before the limit of large-scale separation $\epsilon \to 0$.) The nonlinear large-scale dynamics has been considered by Gama *et al.* (1994) in the two-dimensional case and is beyond the scope of the present paper.

The question of how the time variable should be rescaled depends crucially on the symmetries of the problem. In general, the linear large-scale dynamics is first order in time and first order in space (Frisch *et al.* 1987; Sulem *et al.* 1989). The corresponding physical effect is known as anisotropic kinetic alpha (AKA)-effect. To obtain second order in space large-scale dynamics, it is sufficient to assume parity-invariance, i.e. the presence of a centre of symmetry. An important class of flows not having parity-invariance, but still with no AKA effect, are the *ABC* flows, discussed in §4.2.

Henceforth, we shall assume the absence of an AKA effect. The appropriate rescaling to obtain the 'slow' time variable is then $T = \epsilon^2 t$.

The next step is to use the decomposition rule for time- and space-derivatives:

$$\partial_t \to \partial_t + \epsilon^2 \partial_T, \quad \partial \to \partial + \epsilon \boldsymbol{\nabla}, \tag{5}$$

where $\boldsymbol{\nabla}$ denotes partial derivatives with respect to the slow variable $\boldsymbol{X}$. This reduces the linearized Navier–Stokes problem (4) to a standard singular perturbation problem which can be solved by repeated use of solvability conditions (Fredholm alternatives). The solution is obtained by expanding the large-scale perturbation $(P, \boldsymbol{W})$ in powers of ϵ,

$$\begin{pmatrix} P \\ \boldsymbol{W} \end{pmatrix} = \begin{pmatrix} P^{(0)} \\ \boldsymbol{W}^{(0)} \end{pmatrix} + \epsilon \begin{pmatrix} P^{(1)} \\ \boldsymbol{W}^{(1)} \end{pmatrix} + \epsilon^2 \begin{pmatrix} P^{(2)} \\ \boldsymbol{W}^{(2)} \end{pmatrix} + O(\epsilon^3). \tag{6}$$

To write the equations in a compact form, following DF, it is convenient to introduce an operator notation. The linearized Navier–Stokes equation (4) is rewritten

$$\mathscr{A} \begin{pmatrix} P \\ \boldsymbol{W} \end{pmatrix} \equiv \begin{pmatrix} \mathscr{A}_{PP} & \mathscr{A}_{PW} \\ \mathscr{A}_{WP} & \mathscr{A}_{WW} \end{pmatrix} \begin{pmatrix} P \\ \boldsymbol{W} \end{pmatrix} = 0, \tag{7}$$

where the various matrix blocks are given by

$$\mathscr{A}_{PP} = 0, \quad \mathscr{A}_{PW_i} = \partial_i, \quad \mathscr{A}_{W_i P} = \partial_i, \tag{8}$$

$$\mathscr{A}_{W_i W_j} = (\partial_t - \nu \partial^2)\, \delta_{ij} + \partial_k (\delta_{ij} u_k \bullet + \delta_{kj} u_i \bullet). \tag{9}$$

Here, the bullet symbol $\bullet$ indicates that u_k and u_i act multiplicatively. When the linearized Navier–Stokes operator $\mathscr{A}$ is restricted to functions which have the same space–time periodicity as the force, it will be denoted by A (and similarly from its matrix blocks such as A_{PW}).

Upon use of the decomposition, (4) becomes:

$$(A+\epsilon B+\epsilon^2 C)\begin{pmatrix} P \\ W \end{pmatrix} = 0, \tag{10}$$

where B and C are given by

$$B = \begin{pmatrix} 0 & B_{PW} \\ B_{WP} & B_{WW} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & C_{WW} \end{pmatrix}, \tag{11}$$

with

$$B_{PW_i} = \nabla_i, \quad B_{W_i P} = \nabla_i, \quad B_{W_i W_j} = -2\nu\delta_{ij}\,\partial_k \nabla_k + \nabla_k(\delta_{ij}\, u_k \bullet + \delta_{jk}\, u_i \bullet), \tag{12}$$

$$C_{W_i W_j} = (\partial_T - \nu\nabla^2)\,\delta_{ij}. \tag{13}$$

Since B is linear in slow space derivatives, it may be further decomposed as

$$\left.\begin{aligned} B &= B^l \nabla_l, \\ B^l_{PW_i} = B^l_{W_i P} = \delta_{il}, \quad B^l_{W_i W_j} &= -2\nu\delta_{ij}\,\partial_l + \delta_{ij}\, u_l \bullet + \delta_{jl}\, u_i \bullet. \end{aligned}\right\} \tag{14}$$

One then substitutes (6) into (10) and requires that the result vanishes to all orders in ϵ. The three leading-order equations are:

$$A\begin{pmatrix} P^{(0)} \\ W^{(0)} \end{pmatrix} = 0, \tag{15}$$

$$A\begin{pmatrix} P^{(1)} \\ W^{(1)} \end{pmatrix} + B\begin{pmatrix} P^{(0)} \\ W^{(0)} \end{pmatrix} = 0, \tag{16}$$

$$A\begin{pmatrix} P^{(2)} \\ W^{(2)} \end{pmatrix} + B\begin{pmatrix} P^{(1)} \\ W^{(1)} \end{pmatrix} + C\begin{pmatrix} P^{(0)} \\ W^{(0)} \end{pmatrix} = 0. \tag{17}$$

The assumption of parity-invariance of the basic flow ensures the solvability of (16) (see DF). The solvability condition of (17) gives the desired large-scale equation, namely

$$\nabla_j \langle W_j^{(0)} \rangle = 0, \tag{18}$$

$$\partial_T \langle W_i^{(0)} \rangle = \nu_{ijlm} \nabla_j \nabla_l \langle W_m^{(0)} \rangle - \nabla_i \langle P^{(1)} \rangle. \tag{19}$$

Where $\langle \cdot \rangle$ denotes the space–time average over the periodicities. Thanks to the operator notation, DF have written the 'eddy viscosity tensor' ν_{ijlm} in compact form:

$$\nu_{ijlm} = \nu\delta_{jl}\,\delta_{im} + \langle (B^j \tilde{A}^{-1} B^l (I - \tilde{A}^{-1} A))_{W_i W_m} \rangle. \tag{20}$$

Since in three dimensions the four indices take three values, there may be up to 81 different components of the eddy viscosity tensor. This number may, however, be reduced to 54 by symmetrization of the eddy viscosity tensor in the (j, l) indices, since the latter are contracted with two $\boldsymbol{\nabla}$.

An interesting feature, not stressed in DF, is that the operators appearing in (20) are not acting in the space of divergenceless velocity fields. The reason is that, after use of the decomposition (5), the divergenceless condition will mix different orders of $W^{(n)}$. For example, the divergenceless condition to order ϵ is

$$\boldsymbol{\nabla} \cdot \boldsymbol{W}^{(0)} + \boldsymbol{\partial} \cdot \boldsymbol{W}^{(1)} = 0. \tag{21}$$

This requires some care in the numerical calculation of the eddy viscosities (§3.1).

2.2. *The eddy viscosity as an eigenvalue*

We now show that the concept of eddy viscosity as a number (real or complex) emerges as an eigenvalue problem when studying the system (18)–(19) We may eliminate the pressure $\langle P^{(1)}\rangle$ using the incompressibility condition and the projector

$$P_{ij} \equiv \delta_{ij} - \nabla_i \nabla_j \nabla^{-2}. \tag{22}$$

We thereby obtain from (19)

$$\partial_T \langle W_i^{(0)}\rangle = P_{ir}\, \nu_{rjlm} \nabla_j \nabla_l \langle W_m^{(0)}\rangle. \tag{23}$$

This pseudodifferential equation has plane wave solutions of the form

$$\langle W_i^{(0)}(X,T)\rangle = W_i^{\mathrm{eig}} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{X}) \exp(-\nu_E \boldsymbol{k}^2 T). \tag{24}$$

Substituting (24) into (23) gives us an eigenvalue problem

$$\nu_E \boldsymbol{k}^2 W_i^{\mathrm{eig}} = \nu_{rjlm}\left(\delta_{ir} - \frac{k_i k_r}{\boldsymbol{k}^2}\right) k_l k_j W_m^{\mathrm{eig}}, \tag{25}$$

which may be written as

$$\nu_E \boldsymbol{W}^{\mathrm{eig}} = H(\boldsymbol{k}^0)\, \boldsymbol{W}^{\mathrm{eig}}. \tag{26}$$

Here, $\boldsymbol{k}^0 = \boldsymbol{k}/|\boldsymbol{k}|$, and $H(\boldsymbol{k}^0)$ is the matrix

$$H_{im}(\boldsymbol{k}^0) = \nu_{rjlm}(\delta_{ir} - k_i^0 k_r^0)\, k_j^0 k_l^0. \tag{27}$$

The eddy viscosity ν_E appears thus as an eigenvalue of the matrix $H(\boldsymbol{k}^0)$. Observe that the eigenvalue zero is always present with the corresponding eigenvector parallel to $\boldsymbol{k}^0$. This eigenvalue/eigenvector is, however, not really acceptable since the vector $\boldsymbol{W}^{\mathrm{eig}}$ should be perpendicular to $\boldsymbol{k}$ (by incompressibility). Actually, the matrix H acts non-trivially only in the $(d-1)$-dimensional subspace perpendicular to $\boldsymbol{k}$. Its restriction to this subspace will be denoted $H_\perp$.

It follows that, for a given wave-vector, the eddy viscosities ν_E are the eigenvalues of the $(d-1)\times(d-1)$ real matrix $H_\perp$. The entries of this matrix are real because all the ν_{rjlm} are real.

A consequence is that in two dimensions, the eddy viscosity is always real. This is not so in three dimensions. To examine the question more closely, let us specialize the $\boldsymbol{k}$-vector to the X_3 direction. As long as the basic flow has not been chosen, this is no major restriction. From (25) we obtain the following equation for $\boldsymbol{W}^{\mathrm{eig}} = (W_1^{\mathrm{eig}}, W_2^{\mathrm{eig}}, 0)$

$$\nu_E \begin{pmatrix} W_1^{\mathrm{eig}} \\ W_2^{\mathrm{eig}} \end{pmatrix} = H_\perp \begin{pmatrix} W_1^{\mathrm{eig}} \\ W_2^{\mathrm{eig}} \end{pmatrix} \equiv \begin{pmatrix} \nu_{1331} & \nu_{1332} \\ \nu_{2331} & \nu_{2332} \end{pmatrix} \begin{pmatrix} W_1^{\mathrm{eig}} \\ W_2^{\mathrm{eig}} \end{pmatrix}. \tag{28}$$

Thus, the eddy viscosity eigenvalues are the roots of the quadratic equation

$$\nu_E^2 - (\nu_{1331} + \nu_{2332})\,\nu_E + \nu_{1331}\nu_{2332} - \nu_{1332}\nu_{2331} = 0. \tag{29}$$

If the discriminant

$$\Delta \equiv (\nu_{1331} - \nu_{2332})^2 + 4\nu_{1332}\nu_{2331} \geqslant 0, \tag{30}$$

the eigenvalues are real. In the opposite case ($\Delta < 0$), the eigenvalues are complex (non-real). In §4.1 we shall show that there indeed exist flows such that $\Delta < 0$.

2.3. *The case of cubic symmetry*

It is well known that in three dimensions there exist no crystallographic group ensuring the isotropy of fourth-order tensors (Landau & Lifshitz 1970). With 2π-periodicity in x_1, x_2 and x_3, the best we can have is cubic symmetry, i.e. invariance under coordinate permutations and inversion of coordinates. Note that cubic symmetry singles out a particular frame of reference in which it is convenient to write the eddy viscosity tensor in frame-dependent notation.

The full cubic symmetry group can be generated by the following elements:

$$G_1:(x_1,x_2,x_3)\mapsto(x_1,x_3,-x_2), \tag{31}$$

$$G_2:(x_1,x_2,x_3)\mapsto(x_3,x_2,-x_1), \tag{32}$$

$$P:(x_1,x_2,x_3)\mapsto(-x_1,-x_2,-x_3). \tag{33}$$

Note that P-invariance is just parity-invariance.

It is easy to find examples of incompressible flows with weak cubic symmetry (invariance under rotations of $\frac{1}{2}\pi$ around any of the coordinate axes or translates thereof) which are not parity-invariant. An example is the equilateral ABC-flow discussed in §4.2. An example of incompressible flow invariant under the full cubic symmetry is

$$u_1 = \sin 2x_1 \cos x_2 \cos x_3 - \sin x_1 \cos 2x_2 \cos x_3 - \sin x_1 \cos x_2 \cos 2x_3, \tag{34}$$

$$u_2 = \sin 2x_2 \cos x_3 \cos x_1 - \sin x_2 \cos 2x_3 \cos x_1 - \sin x_2 \cos x_3 \cos 2x_1, \tag{35}$$

$$u_3 = \sin 2x_3 \cos x_1 \cos x_2 - \sin x_3 \cos 2x_1 \cos x_2 - \sin x_3 \cos x_1 \cos 2x_2. \tag{36}$$

Flow which is not parity-invariant will usually give rise to an AKA-effect (although there are exceptions to this, such as Beltrami flows).

As is shown in the Appendix, the most general form of the eddy viscosity tensor with weak cubic symmetry may be taken as

$$\nu_{ijlm} \equiv a\delta_{ijlm} + b(\delta_{im}\delta_{jl} - \delta_{ijlm}), \tag{37}$$

where δ_{ij} and δ_{ijlm} are Kronecker deltas (all indices must be equal for non-vanishing). Note that (37) already holds with just G_i-invariance ($i = 1, 2$). Observe that when $a = b$ the eddy viscosity becomes isotropic. Obviously, when the molecular viscosity is large (low Reynolds number), we have to leading order $a \approx b \approx \nu$.

The expressions of the eddy viscosities (as eigenvalues) for the case of weak cubic symmetry are given in the Appendix.

3. Calculations of eddy viscosities in three dimensions

Equation (20) gives the eddy viscosity in compact form, but not in explicit form, since it involves the inversion of the linearized Navier–Stokes operator $\tilde{A}$, an inversion which can be performed explicitly only in special cases.

The simplest case where everything can be calculated analytically is for layered flow, i.e. when the basic flow depends on a single coordinate, say x_1 (see DF). Such flow, even if it has non-vanishing x_2- and x_3-components, is not genuinely three-dimensional and will not concern us.

Low-Reynolds-number flow, without any restriction on the dimensionality, is amenable to perturbative calculation of the eddy viscosity tensor, as discussed in DF (§III). Their result for the first two terms of the eddy viscosity in an expansion in powers of ν^{-1} will be used subsequently. It reads:

$$\begin{aligned}\nu_{ijlm} = {} & \nu\delta_{im}\delta_{jl} + \nu^{-1}\{-2\langle u_j \mathscr{H}^{-1}\partial^{-2}\partial_i\partial_m u_l\rangle - 2\langle u_i \mathscr{H}^{-1}\partial^{-2}\partial_j\partial_m u_l\rangle \\ & + \langle u_j \mathscr{H}^{-1} u_l\rangle \delta_{im} + \langle u_i \mathscr{H}^{-1} u_l\rangle \delta_{jm} \\ & + 2\langle u_j \mathscr{H}^{-2}\partial_l\partial_m u_i\rangle + 2\langle u_i \mathscr{H}^{-2}\partial_l\partial_m u_j\rangle\} + O(\nu^{-2}).\end{aligned} \tag{38}$$

Here, $\mathscr{H} = \partial_\tau - \partial^2$ is the heat operator and the time τ is related to the fast time by

$$\tau = \nu t. \tag{39}$$

Another instance where the eddy viscosity may be calculated analytically is when the basic flow is random and δ-correlated in time (Gama *et al.* 1994, Appendix D). However, in general the eddy viscosity tensor can be determined only by numerical calculation, as discussed in the next section.

3.1. *Numerical calculations of the eddy viscosity tensor in three dimensions*

Our goal is to compute, for a given basic flow $\boldsymbol{u}$ and a given molecular viscosity ν, all the components of the eddy viscosity tensor given by (20). In what follows we shall limit ourselves to time-independent basic flow. The extension to time-dependence is quite straightforward (see Gama *et al.* 1994 for the two-dimensional case) but can be very taxing in computer resources.

The main problem which must be handled numerically is the calculation of the inverse of the linearized Navier–Stokes operator $\tilde{A}^{-1}$, restricted to functions of zero mean-value. This inverse is well-defined, for large enough molecular viscosities, as long as no eigenvalues of $\tilde{A}$ have crossed the imaginary axis. Otherwise, there will be small-scale instabilities growing on a fast timescale and it becomes uninteresting to study large-scale dynamics.

The space of functions on which act operators such as A, $\tilde{A}^{-1}$ and B are quadruplets of a scalar (the pressure) and of a 3-vector. The latter, as already noted in §2.1, is not restricted to zero divergence.

If a discretization involving N grid points in each direction is used, the discretized quadruplets have thus $4N^3$ degrees of freedom and the operator $\tilde{A}^{-1}$ will be a matrix with $16N^6$ entries. Since N has to be typically 32 or 64 to avoid truncation errors (see below), this is completely unmanageable on present computers. A classical trick is, however, not to store $\tilde{A}^{-1}$, but only the result of its action on successive quadruplet fields. The storage is then reduced to $O(N^3)$ and becomes manageable. As a price, a substantial number of equations of the form

$$\tilde{A}f = g \tag{40}$$

have to be solved. Given the assumption of time-independent basic flow, in all the equations of the form (40), the right-hand side g is time-independent. Note that the linearized Navier–Stokes operator $\tilde{A}$ contains ∂_t but the solution of interest will be time-independent. Still it is convenient to use a time-dependent scheme to integrate (40) and to let the solution relax to convergence.

The numerical calculation is based on a standard (pseudo-) spectral method (Gottlieb & Orszag 1977), which easily allows the use of massively parallel machines, as explained in Gama *et al.* (1994). An $N_1 \times N_2 \times N_3$ regular grid is used with dealiasing by truncation beyond the smallest of the wavenumbers $N_1/3$, $N_2/3$ and $N_3/3$. Periodicity $2\pi\lambda_1$ (respectively, $2\pi\lambda_2$, $2\pi\lambda_3$) is assumed in x_1 (respectively, x_2, x_3). The timestepping is done by a slaved frog scheme (Frisch, She & Thual 1986) with mixing of odd and even solutions every 19 timesteps.

The structure of our code is such that, in d dimensions it allows us to reduce the number of equations of the form $\tilde{A}f = g$ to be solved to $d+d^2$, that is 12 in three dimensions. In contrast, $2d^4$ equations would be needed to get all d^4 entries of the fourth-order eddy-viscosity tensor, calculating one entry after another.

The number of timesteps needed to ensure convergence varies from a few tens to a few thousands, depending on the viscosity. On a CM-200, the CPU requirement (using 8K processors) varies from a few minutes to several hours. The higher resolution (64^3) was used essentially to check that truncation errors are negligible for $\nu \geqslant 1$. Results from 32^3 and 64^3 calculations agree to at least five decimal places.

The code has been tested in various ways. First, we tested it on layered flow for which an analytic expression of the eddy viscosity tensor is known (DF). Since an already well-tested code exists for two-dimensional flow (Gama *et al.* 1994), we used it to test our three-dimensional code, successively assuming a flow which has no dependence on the x_1-, on the x_2- and on the x_3-coordinate. Finally, we took advantage of the existence of the low-Reynolds-number expansion (38) to perform further tests.

4. Applications

4.1. *Search for flows with complex eddy viscosity*

As seen in §2.2, to obtain a complex (non-real) eddy viscosity we must have $\Delta < 0$, where Δ is the discriminant given by (30). It is shown in the Appendix that flow with cubic symmetry has $\Delta > 0$. Do flows exist with the property $\Delta < 0$?

We started a search based on the perturbation expansion (38) of the eddy viscosity tensor. The only components appearing in Δ are:

$$\nu_{1331} - \nu_{2332} = \frac{2}{\nu}\{\langle u_3 \partial^{-4}(\partial_1\partial_1 - \partial_2\partial_2)u_3\rangle + 3\langle u_3\partial^{-4}\partial_3(\partial_1 u_1 - \partial_2 u_2)\rangle\} + O(\nu^{-2}), \tag{41}$$

$$\nu_{1332} = \frac{2}{\nu}\{\langle u_3\partial^{-4}\partial_1\partial_2 u_3\rangle + 3\langle u_3\partial^{-4}\partial_2\partial_3 u_1\rangle\} + O(\nu^{-2}), \tag{42}$$

$$\nu_{2331} = \frac{2}{\nu}\{\langle u_3\partial^{-4}\partial_2\partial_1 u_3\rangle + 3\langle u_3\partial^{-4}\partial_1\partial_3 u_2\rangle\} + O(\nu^{-2}). \tag{43}$$

We now want to find a flow such that $\Delta \equiv (\nu_{1331}-\nu_{2332})^2 + 4\nu_{1332}\nu_{2331} < 0$. A necessary condition for this is that ν_{1332} and ν_{2331} should have opposite signs. The first terms on the right-hand side in ν_{1332} and ν_{2331} are the same and the second will be the same if the flow is invariant under interchange of x_1 and x_2. Furthermore, the difference between ν_{1332} and ν_{2331} involves $\delta = \langle(\partial_1 u_2 - \partial_2 u_1)\partial^{-2}\partial_3 u_3\rangle$. Note that the quantity $\partial_1 u_2 - \partial_2 u_1$ is the x_3-component of the vorticity. If the velocity and the vorticity only differ by a multiplicative constant (as is the case for the *ABC* flows), then δ will be proportional to $\langle u_3\partial^{-2}\partial_3 u_3\rangle$ which vanishes. Clearly, we want to avoid situations where ν_{1332} and ν_{2331} are equal, since this implies $\Delta \geqslant 0$. We found a flow for which ν_{1332} and ν_{2331} are actually of different signs, namely $\boldsymbol{u} = (\sin(x_1+x_2-x_3), 0, \sin(x_1+x_2-x_3))$. This flow does not have $\Delta < 0$, because the $(\nu_{1331}-\nu_{2332})^2$ term overwhelms the negative term $\nu_{1332}\nu_{2331}$. We now observe that by adding a suitable term proportional to $\sin x_1$ to the x_3-component of the velocity, we can cancel the leading-order contribution to $(\nu_{1331}-\nu_{2332})^2$ without affecting $\nu_{1332}\nu_{2331}$. We are thus led to the following basic flow:

$$\boldsymbol{u} = (\sin(x_1+x_2-x_3), 0, \sin(x_1+x_2-x_3) + (1/\surd 3)\sin x_1), \tag{44}$$

for which

$$\Delta = -\tfrac{2}{81}\nu^2 + O(\nu^{-3}). \tag{45}$$

This establishes the existence of a flow with complex eddy viscosities:

$$\nu_E = \nu + \frac{1}{\nu}\left(\frac{2}{9} \pm \frac{\mathrm{i}\surd 2}{9}\right) + O(\nu^{-2}). \tag{46}$$

Note that the flow (44) is two-and-a-half-dimensional in the sense that it depends only on two coordinates, namely x_1 and $x' = x_2 - x_3$. The fact that strictly two-dimensional flow cannot have a complex eddy viscosity does not carry over to two-and-a-half dimensional flow. Indeed, the latter is subject to a large-scale perturbation which depends not just on coordinates in the (x_1, x')-plane.

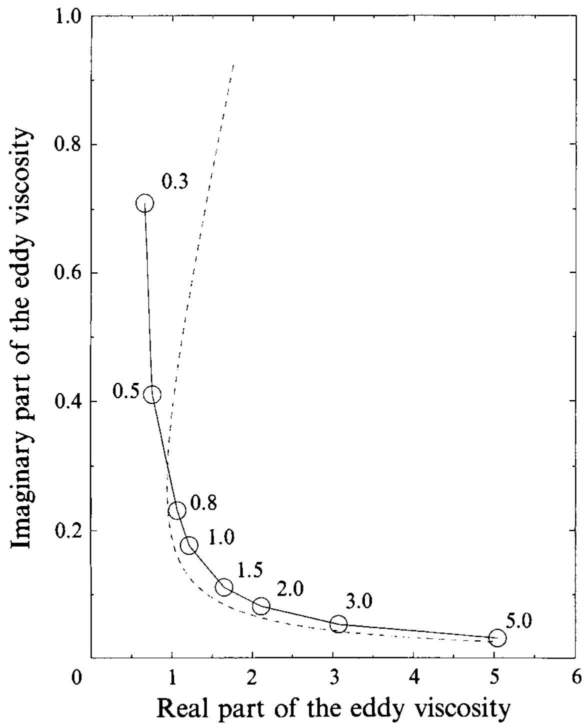

FIGURE 1. Real and imaginary parts of the eddy viscosity eigenvalue for the two-and-a-half-dimensional flow (44). The labels near the circles (computed values) show the molecular viscosities. ---, first two terms from the low-Reynolds-number expansion (46).

We have thus shown that the two-and-a-half-dimensional flow has a complex eddy viscosity for arbitrarily small Reynolds numbers. However, the calculation being perturbative, the imaginary part in (46) is much smaller than the real part. Hence, dispersive effects are dominated by diffusion. To avoid this, we must drop the assumption of very low Reynolds numbers and make numerical calculations using the method of §3.1.

Figure 1 shows the computed real and imaginary parts for the eddy viscosity of the two-and-a-half-dimensional flow, together with the perturbative expression (38), for a range of values of the molecular viscosity (as labelled).

At high molecular viscosities (say, higher than 20), not shown on the figure, there is almost perfect agreement with the perturbative expression. As the viscosity is lowered, the low-Reynolds-number expression deteriorates, as expected. It is noteworthy that for the smallest viscosity used, $\nu = 0.3$, the imaginary part is already larger than the real part, so that strong dispersive effects are present.

Another instance of a flow with complex eddy viscosity has been brought to our attention by A. A. Nepomnyashchy and Y. Hazan (private communication).

4.2. *Negative eddy viscosity instability for the ABC-flow*

The *ABC*-flows are the 3-parameter family of flows, 2π periodic in x_1, x_2 and x_3, defined by

$$u_1 = A\sin x_3 + C\cos x_2, \tag{47}$$

$$u_2 = B\sin x_1 + A\cos x_3, \tag{48}$$

$$u_3 = C\sin x_2 + B\cos x_1. \tag{49}$$

They were introduced by Arnold (1965).† Because of their use in the study of chaotic advection and the dynamo effect, there is considerable literature on these flows (see e.g. Dombre *et al.* 1986; Galloway & Frisch 1987 and references therein).

What concerns us here is the stability of *ABC*-flows, considered as solutions of the Navier–Stokes equation with a forcing of the *ABC*-type: when $\boldsymbol{u}$ is of the *ABC*-type the nonlinear term vanishes so that the force and the viscous term balance.

Although we studied several instances of the *ABC*-flows, we shall report numerical results only for the 'equilateral' case which has maximum symmetry:

$$A = B = C = 1. \tag{50}$$

It is then customary to define the Reynolds number R as the inverse of the viscosity.

Galloway & Frisch (1987) have found that the equilateral *ABC*-flow, when subject to weak perturbations having the same 2π-periodicity as the basic flow, becomes unstable at a critical Reynolds number just above 13. (This has been confirmed by Zheligovsky & Pouquet (1993), who have also studied the nature of the ensuing bifurcation.)

We shall now show that when the perturbations are allowed to have a scale much larger than that of the basic flow, the critical Reynolds number for instability may drop by almost one order of magnitude.

The *ABC*-flows are not parity-invariant. Actually, they possess a non-vanishing helicity $\int \boldsymbol{u}\cdot\boldsymbol{\nabla}\wedge\boldsymbol{u}\,\mathrm{d}^3x$. Still, the fact that they possess the Beltrami property, ($\boldsymbol{\nabla}\wedge\boldsymbol{u}$ and $\boldsymbol{u}$ are equal) ensures the absence of an AKA-effect. Technically, this means that the solvability condition for (16) is satisfied. Our eddy viscosity formalism is thus directly applicable to *ABC*-flows.

For arbitrary periodic flow, the null-space of the linearized Navier–Stokes operator, i.e. the solution of (15), must be determined numerically. For *ABC*-flows the null-space may be obtained analytically. Indeed, it may be easily checked that in the four-dimensional notation of §2.1 (pressure followed by the three components of the velocity), the null-space is a three-dimensional vector-space spanned by:

$$\boldsymbol{N}_1 = \begin{pmatrix} BC\cos x_1\cos x_2 - AB\sin x_1\sin x_3 \\ -\nu \\ -B\sin x_1 \\ B\cos x_1 \end{pmatrix}, \tag{51}$$

$$\boldsymbol{N}_2 = \begin{pmatrix} AC\cos x_2\cos x_3 - BC\sin x_2\sin x_1 \\ C\cos x_2 \\ -\nu \\ -C\sin x_2 \end{pmatrix}, \tag{52}$$

$$\boldsymbol{N}_3 = \begin{pmatrix} AB\cos x_3\cos x_1 - AC\sin x_3\sin x_2 \\ -A\sin x_3 \\ A\cos x_3 \\ -\nu \end{pmatrix}. \tag{53}$$

From now on we restrict ourselves to the equilateral *ABC*-flow. This flow has weak cubic symmetry. Its eddy viscosity tensor has the general form (37) which involves only

† Equations (47) and (48) are already found in Beltrami (1889: see p. 304 of the complete works) but with the coefficients *ABC* (denoted T_1, T_2, T_3) having 'an arbitrary time-dependence'. Note that, in general, the time-dependence prevents such flows from being solutions of the steady-state Euler equation.

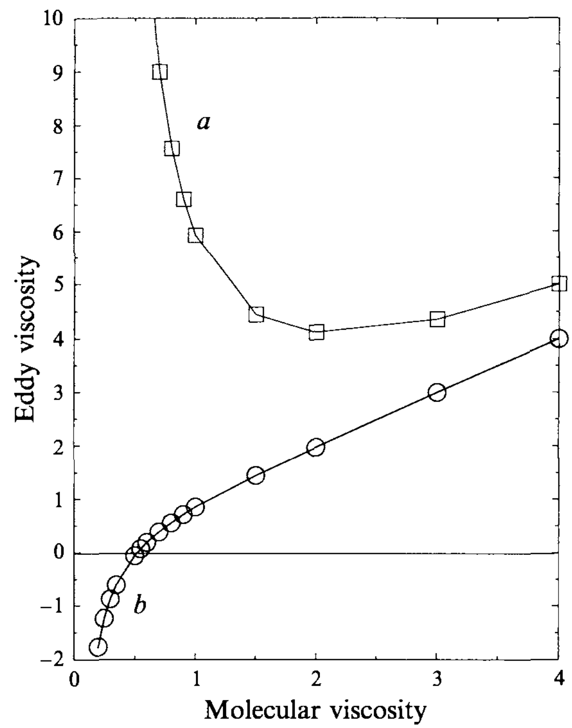

FIGURE 2. The two basic parameters in the eddy viscosity tensor (37) for the equilateral ABC-flow.

two scalars $a(\nu)$ and $b(\nu)$. The two eddy viscosities (eigenvalues), corresponding to a given wave vector $\boldsymbol{k} = (k_1, k_2, k_3)$ are given explicitly in the Appendix (equation (A 2)).

In the Appendix it is also shown that, for any flow having the weak cubic symmetry, the eddy viscosities (eigenvalues) are real. Furthermore, it is shown that when $a > 0$ and $b > 0$ the eddy viscosities are positive (stability). Finally, it is shown that for $a > 0$ and when b crosses over to negative values, at least one of the eddy viscosities becomes negative (instability), for those wave vectors in the three planes generated by any pair of coordinate axes.

The determination of $a(\nu)$ and $b(\nu)$ was done by the numerical technique presented in §3. Figure 2 shows the variation of the coefficients $a(\nu)$ and $b(\nu)$ for a range of values of the molecular viscosity. It is seen that $a(\nu) > 0$. Hence, the stability of the equilateral ABC-flow is governed solely by the sign of the coefficient $b(\nu)$. The latter changes sign when ν crosses the value $\nu_c \simeq 0.52$.

Thus, for $\nu < \nu_c$ a negative-viscosity large-scale instability appears. The corresponding Reynolds number is

$$R_c = \frac{1}{\nu_c} = 1.92. \tag{54}$$

This value is nearly seven times smaller than the critical value obtained by Galloway & Frisch (1987) for instability to perturbations having the same 2π spatial periodicity as the basic flow.

The nature of the modes which become unstable at $R = R_c$ can be inferred from our knowledge of the null-space of the linearized Navier–Stokes operator. Let us assume for example, that the leading-order term $W^{(0)}$ has an $\exp(ik_1 X_1)$ dependence on the slow coordinates (with $k_1 \neq 0$). From (51)–(53), specialized to $A = B = C = 1$, it follows that $W^{(0)}$ has the following representation

$$W^{(0)} = \exp(ik_1 X_1)\left[\lambda_1 \begin{pmatrix} -\nu \\ -\sin x_1 \\ \cos x_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} \cos x_2 \\ -\nu \\ -\sin x_2 \end{pmatrix} + \lambda_3 \begin{pmatrix} \sin x_3 \\ \cos x_3 \\ -\nu \end{pmatrix}\right]. \tag{55}$$

Taking the fast average of (21), we obtain $\nabla \cdot \langle W^{(0)} \rangle = 0$. This implies that $\lambda_1 = 0$, while λ_2 and λ_3 are arbitrary.

It is of interest to compare the negative-viscosity instability of the equilateral *ABC*-flow to the well-known instance of the Kolmogorov flow which has a negative-viscosity instability for $\nu = \nu_c = 1/\sqrt{2}$ (see e.g. Meshalkin & Sinai 1961 and DF). In the *ABC* flow, when $A = 1$ and $B = C = 0$, a circularly polarized one-dimensional flow is obtained. It is easily checked that its stability properties are just the same as for the Kolmogorov flow. We may define an effective Reynolds number

$$R_{\text{eff}} = \frac{Lv}{\nu}, \tag{56}$$

where L is the inverse of the wavenumber in the basic flow (always one, here) and v is the r.m.s. velocity of the basic flow. For the Kolmogorov flow, we have $R_{\text{eff}} = 1/(\nu\sqrt{2})$, while for the equilateral *ABC*-flow we have $R_{\text{eff}} = \sqrt{3}/\nu$. In terms of this effective Reynolds number, the critical value is more than three times higher for the equilateral *ABC*-flow than for the Kolmogorov flow.

For the Kolmogorov flow the nonlinear regime, just beyond the critical value, has been studied by Nepomnyashchy (1976), Sivashinsky (1985), She (1987) and others. In principle, a similar study is feasible for the equilateral *ABC*-flow. It is, however, expected to be much more involved, since the bifurcation is highly degenerated: for any wave-vector in a plane spanned by two coordinate axes, there is zero-crossing of an eigenvalue at the critical viscosity ν_c.

Let us finally consider the thermodynamics of negative eddy viscosity. It is generally believed that molecular viscosities are constrained by thermodynamics to be positive and, so, negative eddy viscosities could be violating thermodynamics. The true situation is considerably more subtle. First, there is a basic difference between fluids considered microscopically, which are conservative dynamical systems and the same fluids, described macroscopically which are governed by dissipative equations. In the latter case, there is no such thing as a thermodynamic equilibrium. There may be non-trivial attractors for long times, but only if the system is driven in some way, for example, through external forces, pressure gradients or thermal convection. When negative eddy viscosity instabilities are present in the large-scale dynamics, the flow may exhibit new long-range orders, but no violation of microscopic thermodynamics is happening.

Secondly, negative viscosity phenomena can actually occur in certain microscopic conservative systems. Rothman (1989) and Hénon (1992) have shown that some lattice gases in which the condition of semi-detailed-balance (a weak form of microreversibility) is released, can display negative viscosities. This requires a fine-tuning of the collision laws which is akin to introducing a Maxwell demon. The large-scale instabilities which are obtained in flows such as the Kolmogorov flow or the *ABC*-flows are of a very different nature: they are just a particular instance of hydrodynamic instability.

5. The issue of isotropy

Can we find flows having an isotropic eddy viscosity, that is, such that the tensor ν_{ijlm} is given by (37) with $a = b$? There is one trivial instance, namely when the flow vanishes and the eddy viscosity reduces to the molecular viscosity. We also know that in two dimensions, spatially periodic flow with suitable symmetry has an isotropic eddy viscosity (Vergassola, Gama & Frisch 1994; Gama *et al.* 1994).

We have already stressed that in three dimensions isotropy is generically not consistent with periodicity. It may be that among the flows with cubic symmetry some have accidental isotropy (i.e. $a = b$ in equation (A 2)) for certain values of the control parameters.

A more generic way to ensure isotropy is to work with random, homogeneous and isotropic flows rather than with deterministic ones. Formally, the theory of the eddy viscosity can be extended to this case. There are however serious mathematical and numerical difficulties. For example, the eddy viscosity must be calculated by ensemble averaging over a large number of realizations (Monte Carlo method). This may require very large computer resources.

A particularly appealing alternative is to work with quasi-periodic flow having icosahedral symmetry, as suggested by Yakhot, Bayly & Orszag (1986). A simple example is the flow generated by superposing 12 plane waves of equal amplitudes having the wave vectors

$$\begin{pmatrix} \pm 1 \\ \pm\tau \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \pm 1 \\ \pm\tau \end{pmatrix}, \quad \begin{pmatrix} \pm\tau \\ 0 \\ \pm 1 \end{pmatrix}. \tag{57}$$

Here $\tau = \frac{1}{2}(1+\surd 5)$ is the golden mean. Such icosahedral symmetry guarantees isotropy of tensors up to fourth order. With quasi-periodic flow, a number of new challenges appear. Equations such as (15)–(17) become linear partial differential equations with quasi-periodic coefficients. Much is known about the wave or Schrödinger equations in one dimension with quasi-periodic potential; for example, the possibility of Bloch-type or localized solutions (see e.g. Chulaevsky 1989; Pastur & Figotin 1992). The linearized Navier–Stokes operator (known as the Orr–Sommerfeld operator when the basic flow depends on a single coordinate) is, however, not self-adjoint, contrary to the Schrödinger operator, so that existing theory cannot be easily adapted.

For the numerical solution of the linearized Navier–Stokes with quasi-periodic coefficients, we have explored a new method based on multiplying the dimension of the space by the number of incommensurate wavenumbers in the basic flow (two for the case of (57)). The method seems to work well in two dimensions (where it requires the use of a four-dimensional spectral code). In three dimensions, the possibilities are severely restricted by the present limitation of even the fastest machines.

Finally, there is the issue of having an eddy viscosity, which is simultaneously isotropic and negative. The equilateral *ABC*-flow has a negative eddy viscosity, but only for special directions of the wave vector. In two-dimensions, we know that negative isotropic eddy viscosity is a rather common phenomenon (Gama *et al.* 1994). This phenomenon may also occur in three dimensions, but is likely to be less common. One indication is that when the basic flow is random isotropic and time-dependent with a very short correlation-time, the contribution to the molecular viscosity vanishes in two dimensions but is always positive in three dimensions (Gama *et al.* 1994, Appendix D).

Furthermore, in three dimensions there is no known equivalent to the inverse cascade (Kraichnan 1967) so that we do not 'need' negative isotropic eddy viscosity. It may exist nevertheless.

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Appendix. Eddy viscosity for cubic symmetry

The highest symmetry in three dimensions compatible with periodicity is cubic symmetry, i.e. invariance under parity and permutations of the coordinates x_1, x_2 and x_3.

Actually, we shall need only weak cubic symmetry, i.e. invariance under rotations of angle $\frac{1}{2}\pi$ around any of the coordinate axes or translates thereof. The basic equations for the large-scale perturbation, given in §2.1, are (18) and (19), which involve the eddy viscosity tensor ν_{ijlm} in the combination $\nabla_j \nabla_l \langle W_m^{(0)} \rangle$. Without loss of generality, we may thus assume that ν_{ijlm} is symmetrical in (j, l). Using the same technique as in Landau & Lifshitz (1970, p. 54), and slightly different notation, we can see that the fourth-order tensor ν_{ijlm} with such symmetry in its indices and consistent with weak cubic symmetry depends only on three real numbers a, b and c. In the (x_1, x_2, x_3)-frame we can write the components of the eddy viscosity tensor in the following form:

$$\nu_{ijlm} \equiv a\delta_{ijlm} + b(\delta_{im}\delta_{jl} - \delta_{ijlm}) + c(\delta_{il}\delta_{jm} + \delta_{ij}\delta_{lm}), \tag{A 1}$$

where δ_{ijlm} is equal to one if all four indices are equal and to zero otherwise; this is of course not a coordinate-independent representation of the eddy viscosity tensor, because δ_{ijlm} is not a tensor. Since in (19) the eddy viscosity tensor is contracted with $\nabla_j \nabla_l$ and $\nabla_m \langle W_m^{(0)} \rangle = 0$, the last term, involving the coefficient c, does not contribute. Hence, no generality is lost by assuming $c = 0$. Using now (A 1) in the definition (27) of the matrix $H_{im}(\boldsymbol{k}^0)$, we can express the latter in terms of the parameters a and b and the direction $\boldsymbol{k}^0$ of the wave vector. The resulting three-by-three matrix has entries depending on five parameters: the viscosity, the coefficients a and b, and the direction of the wave vector $\boldsymbol{k}_0$. The determination of its non-vanishing eigenvalues may be done for example using symbolic calculations. The final result for the eddy viscosities reads:

$$\nu_E^{\pm} = \frac{a+b}{k^4}(k_1^2 k_2^2 + k_1^2 k_3^2 + k_2^2 k_3^2) + \frac{b}{k^4}(k_1^4 + k_2^4 + k_3^4) \pm \frac{a-b}{k^4} \Delta^{1/2}, \tag{A 2}$$

where

$$\Delta \equiv k_1^4 k_2^4 + k_1^4 k_3^4 + k_2^4 k_3^4 - k_1^4 k_2^2 k_3^2 - k_1^2 k_2^4 k_3^2 - k_1^2 k_2^2 k_3^4, \tag{A 3}$$

and $k^4 \equiv (k_1^2 + k_2^2 + k_3^2)^2$. We show now that Δ is always positive. Δ being homogeneous and completely symmetrical in (k_1, k_2, k_3), without loss of generality, we may assume $k_1^2 = 1 \geqslant k_2^2 = \sigma \geqslant k_3^2 = \tau \geqslant 0$. We thus obtain

$$\Delta = \sigma^2 + \tau^2 + \sigma\tau((\tau - 1)\sigma - \tau - 1). \tag{A 4}$$

We now observe that, from $0 \leqslant \tau \leqslant \sigma \leqslant 1$, it follows that $(\tau - 1)\sigma - \tau - 1 \geqslant$

$(\tau-1)-\tau-1 = -2$. Thus, $\Delta \geqslant \sigma^2+\tau^2-2\sigma\tau = (\sigma-\tau)^2 \geqslant 0$. The positivity of Δ implies that ν_E^+ and ν_E^- are both real. We have thus shown that weak cubic symmetry implies always real eddy viscosity.

Next, we shall show that $a > 0$ and $b > 0$ implies $\nu_E^\pm > 0$. We can rewrite (A 2) as

$$\nu_E^\pm = \frac{a}{k^4}(\Theta \pm \Delta^{1/2}) + \frac{b}{k^4}(\Theta + \Lambda \mp \Delta^{1/2}), \quad \text{(A 5)}$$

where

$$\Theta = (k_1^2 k_2^2 + k_1^2 k_3^2 + k_2^2 k_3^2), \quad \text{(A 6)}$$

and

$$\Lambda = (k_1^4 + k_2^4 + k_3^4). \quad \text{(A 7)}$$

We observe that

$$\Theta \geqslant \Delta, \quad \text{(A 8)}$$

equality being obtained only when all cross-products (involving all three factors k_1, k_2 and k_3) vanish. In other words, equality holds only when the vector $\boldsymbol{k}$ is in a coordinate plane, i.e. is in any of the three planes generated by pairs of coordinate axes. From this, it follows that the first term in (A 5) is never negative, and that the second is strictly positive (by the fact that $\Lambda > 0$), so that $\nu_E^\pm > 0$.

Finally, we show that for $a > 0$, when b crosses the value zero, at least one of the eddy viscosities becomes negative for those $\boldsymbol{k}$ in the coordinate planes and otherwise not. For $\boldsymbol{k}$ in a coordinate plane, we have $\Theta = \Delta$. Hence, for $a > 0$ and b slightly negative, the smallest eddy viscosity is $\nu_E^- = b(2\Theta+\Lambda)/k^4 = b$, which changes sign with b. Actually, if $\boldsymbol{k}$ is parallel to one of the coordinate axes, $\nu_E^+ = \nu_E^- = b$ and the eddy viscosities both change sign with b. When $\boldsymbol{k}$ is not in a coordinate plane, the first term in (A 5) is strictly positive and no change of sign occurs when b goes through zero.

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